Classification of real three-dimensional Lie bialgebras and their Poisson-Lie groups

A. Rezaei-Aghdam [*], M. Hemmati and A.R. Rastkar

Department of Physics, Azarbaijan University of Tarbiat Moallem, 53714-161, Tabriz, Iran .

February 7, 2008

Abstract

Classical r-matrices of the three-dimensional real Lie bialgebras are obtained. In this way all three-dimensional real coboundary Lie bialgebras and their types (triangular, quasitriangular or factorizable) are classified. Then, by using the Sklyanin bracket, the Poisson structures on the related Poisson-Lie groups are obtained.

[*]e-mail:rezaei-a@azaruniv.edu

1 Introduction

As is well known by now, the theory of classical integrable systems is naturally related to the geometry and representation theory of Poisson-Lie groups and the corresponding Lie bialgebras [1] and their classical r-matrices [2](see for example [3] and [4]). Of course recently Lie bialgebras and their Poisson-Lie groups have application in the theory of Poisson-Lie T-dual sigma models [5]. Up to now there is a detailed classification of r-matrices only for the complex semi-simple Lie algebras [6]. On the other hand, recently non-semisimple Lie algebras have important role in the physical problems. Of course there are attempts for the classification of low dimensional Lie bialgebras $[7-11]$. In ref [7], the classification of complex three dimensional Manin pairs related to the complex three dimensional Lie algebras has been performed and in this way by use of the connection between Manin triples and the $N=2$ superconformal field theory [13], all $N=2$ structures with $c=9$ has been classified. In [9] and [10], by use of mixed Jacobi identity for bialgebras the authors obtain all three dimensional Lie bialgebras. Classification of the complex and real three dimensional Lie bialgebras have been performed in [11] on the same footing by using extensively the notion of twisting due to V. G. Drinfeld [1]. In this manner three dimensional real coboundary Lie bialgebras are obtianed. In Ref [11] the classification of three dimensional Lie algebras of ref [12] has been applied. On the other hand in physical models the Bianchi classification of three dimensional Lie algebras [16] are applied. In Refs [9] and [10] and other applications of them [14], this classification has been applied. On the other hand in [9] and [10] the type of Lie bialgebras (coboundary or not) have not been recognized. In this paper we perform this and classify all three dimensional real coboundary Lie bialgebras and determine their types (triangular or quasitriangular). Furthermore we calculate Poisson structures on the corresponding Poisson-Lie groups. In this way, one is ready to perform the quantization of these Lie bialgebras.

The paper is organized as follows. In section two, we recall some basic definitions and propositions, then review how to obtain the three dimensional real Lie bialgebras [9] and [10]. By calculating and use of automorphism groups of Bianchi algebras we show that these Lie bialgebras are nonisomorphic. In section three, we determine types of 44 Lie bialgebras i.e are these coboundary (triangular or quasitriangular) or not? We list coboundary Lie biagebras in tables 3 and 4. We list coboundary Lie bialgebras with coboundary duals in a separate table 4. At the end of this section we show that these coboundary Lie bialgebras are nonisomorphic. Finally, in section four we calculate Poisson structures on the Poisson-Lie groups by using of the Sklyanin bracket.

2 Three dimensional real Lie bialgebras

Let us recall some basic definitions and propositions [1], [3], [4]. Let $\mathbf{g}$ be a finite-dimensional Lie algebra and $\mathbf{g}^*$ be its dual space with respect to a non-degenerate canonical pairing $(,)$ on $\mathbf{g}^* \times \mathbf{g}$.

Definition: A *Lie bialgebra* structure on a Lie algebra $\mathbf{g}$ is a skew-symmetric linear map $\delta : \mathbf{g} \longrightarrow \mathbf{g} \otimes \mathbf{g}$ (*the cocommutator*) such that:

a)δ is a one-cocycle, i.e.:

$$\delta([X,Y]) = [\delta(X), 1\otimes Y + Y\otimes 1] + [1\otimes X + X\otimes 1, \delta(Y)] \qquad \forall X, Y \in \mathbf{g}. \tag{1}$$

b) The dual map $\delta^t : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ is a Lie bracket on $\mathbf{g}^*$:

$$(\xi\otimes\eta, \delta(X)) = (\delta^t(\xi\otimes\eta), X) = ([\xi,\eta]_*, X) \qquad \forall X \in \mathbf{g};\ \ \xi,\eta \in \mathbf{g}^*. \tag{2}$$

The Lie bialgebra defined in this way will be denoted by $(\mathbf{g}, \mathbf{g}^*)$ or $(\mathbf{g}, \delta)$. Notice that the notation $(\mathbf{g}, \mathbf{g}^*)$ is less precise since as we will see there might be several nonequivalent one-cocycles on $\mathbf{g}$ giving isomorphic Lie algebra structures to $\mathbf{g}^*$, however because of consistency and application of the results of Refs [9], [10] we will consider the notions $(\mathbf{g}, \mathbf{g}^*)$.

Proposition: One-cocycles δ and δ' of the algebra $\mathbf{g}$ are said to be *equivalent* if there exists an automorphism O of $\mathbf{g}$ such that:

$$\delta' = (O\otimes O)\circ\delta\circ O^{-1}. \tag{3}$$

In this case two Lie bialgebras $(\mathbf{g}, \delta)$ and $(\mathbf{g}, \delta')$ are equivalent [3], [4].

Definition:A Lie bialgebra is called *coboundary* Lie bialgebra if the cocommutator is a one-coboundary, i.e, if there exist an element $r \in \mathbf{g}\otimes\mathbf{g}$ such that:

$$\delta(X) = [1\otimes X + X\otimes 1, r] \qquad \forall X \in \mathbf{g}. \tag{4}$$

Proposition: Two coboundary Lie bialgebras $(\mathbf{g}, \mathbf{g}^*)$ and $(\mathbf{g}', \mathbf{g}^{*\prime})$ defined by $r \in \mathbf{g}\otimes\mathbf{g}$ and $r' \in \mathbf{g}'\otimes\mathbf{g}'$ are *isomorphic* if and only if there is an isomorphism of Lie algebras $\alpha : \mathbf{g} \longrightarrow \mathbf{g}'$ such that $(\alpha\otimes\alpha)r - r'$ is $\mathbf{g}'$ invariant i.e:

$$[1\otimes X + X\otimes 1, (\alpha\otimes\alpha)r - r'] = 0 \qquad \forall X \in \mathbf{g}'. \tag{5}$$

Definition:Coboundary Lie bialgebras can be of two different types:

a) If r is a skew-symmetric solution of the classical Yang-Baxter equation (CYBE):

$$[[r, r]] = 0, \tag{6}$$

then the couboundary Lie bialgebra is said to be *triangular*; where in the above equation the schouten bracket is defined by:

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{7}$$

and if we denote $r = r^{ij} X_i \otimes X_j$, then $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$. A solution of the CYBE is often called a *classical r-matrix*.

b) If r is a solution of CYBE, such that $r_{12} + r_{21}$ is a $\mathbf{g}$ invariant element of $\mathbf{g}\otimes\mathbf{g}$; then the coboundary Lie bialgebra is said to be *quasi-triangular*. If moreover, the symmetric part of r is invertible, then r is called *factorizable*.

Sometimes condition *b*) can be replaced with the following one [1],[3]:

$b')$ If r is a skew-symmetric solution of the modified CYBE :

$$[[r,r]] = \omega \qquad \omega \in \wedge^3 \mathbf{g}, \tag{8}$$

then the coboundary Lie bialgebra is said to be quasi-triangular.

Notice that if $\mathbf{g}$ is a Lie bialgebra then $\mathbf{g}^*$ is also a Lie bialgebra [3] but this is not always true for the coboundary property.

Definition: Suppose that $\mathbf{g}$ be a coboundary Lie bialgebra with one-coboundary (4); and furthermore suppose that $\mathbf{g}^*$ is also coboundary Lie bialgebra with the one-coboundary:

$$\forall \xi \in \mathbf{g}^* \qquad \exists r^* \in \mathbf{g}^* \otimes \mathbf{g}^* \qquad \delta^*(\xi) = [1 \otimes \xi + \xi \otimes 1, r^*]_*, \tag{9}$$

where $\delta^* : \mathbf{g}^* \longrightarrow \mathbf{g}^* \otimes \mathbf{g}^*$. Then the pair $(\mathbf{g}, \mathbf{g}^*)$ is called a bi-r-matrix bialgebra [15]if the Lie bracket $[,]'$ on $\mathbf{g}$ defined by δ^{*t}:

$$(\delta^*(\xi), X \otimes Y) = (\xi, \delta^{*t}(X \otimes Y)) = (\xi, [X,Y]') \qquad \forall X, Y \in \mathbf{g}, \qquad \xi \in \mathbf{g}^*, \tag{10}$$

is equivalent to the original ones [15]:

$$[X,Y]' = S^{-1}[SX, SY] \qquad \forall X, Y \in \mathbf{g}, \qquad S \in Aut(\mathbf{g}). \tag{11}$$

Definition A *Manin* triple is a triple of Lie algebras $(\mathcal{D}, \mathbf{g}, \tilde{\mathbf{g}})$ together with a non-degenerate ad-invariant symmetric bilinear from $<,>$ on $\mathcal{D}$ such that;

a)$\mathbf{g}$ and $\tilde{\mathbf{g}}$ are Lie subalgebras of $\mathcal{D}$,

b) $\mathcal{D} = \mathbf{g} \otimes \tilde{\mathbf{g}}$ as a vector space

c) $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isotropic with respect to $<,>$, i.e:

$$< X_i, X_j > = < \tilde{X}^i, \tilde{X}^j > = 0, \qquad\qquad < X_i, \tilde{X}^j > = \delta_i{}^j, \tag{12}$$

where $\{X_i\}$ and $\{\tilde{X}^i\}$ are the bases of the Lie algebras $\mathbf{g}$ and $\tilde{\mathbf{g}}$, respectively. There is a one-to-one correspondence between Lie bialgebra $(\mathbf{g}, \mathbf{g}^\star)$ and Manin triple $(\mathcal{D}, \mathbf{g}, \tilde{\mathbf{g}})$ with $\tilde{\mathbf{g}} = \mathbf{g}^\star$ [3], [4]. If we choose the structure constants of algebra $\mathbf{g}$ and $\tilde{\mathbf{g}}$ as follows:

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad\qquad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k, \tag{13}$$

then ad-invariance of the bilinear form $<,>$ on $\mathcal{D} = \mathbf{g} \otimes \tilde{\mathbf{g}}$ implies that [3]:

$$[X_i, \tilde{X}^j] = \tilde{f}^{jk}{}_i X_k + f_{ki}{}^j \tilde{X}^k. \tag{14}$$

Clearly by use of the equations (12), (13) and (2) we have:

$$\delta(X_i) = \tilde{f}^{jk}{}_i X_j \otimes X_k. \tag{15}$$

By applying this relation in the one-cocycle condition (1) one can obtain the following relation [1]:

$$f_{mk}{}^i \tilde{f}^{jm}{}_l - f_{ml}{}^i \tilde{f}^{jm}{}_k - f_{mk}{}^j \tilde{f}^{im}{}_l + f_{ml}{}^j \tilde{f}^{im}{}_k = f_{kl}{}^m \tilde{f}^{ij}{}_m. \tag{16}$$

[1] The above relation can also be obtained from mixed Jacobi identity for (14).

In some literature the above relation is used to the definition of Lie bialgebras.

Now by reviewing these definitions and propositions we are ready to review the works about three dimensional real Lie bialgebras. In fact, in [9] we had applied the above relations for obtaining 28 real three dimensional Bianchi bialgebra (Lie bialgebras where its duals are of Bianchi type). In so doing, we had considered the Behr's classification of three dimensional Bianchi Lie algebras [16], as follows:

$$[X_1, X_2] = -aX_2 + n_3X_3, \qquad [X_2, X_3] = n_1X_1,$$
$$[X_3, X_1] = n_2X_2 + aX_3, \tag{17}$$

where the structure constants are given in table 1.

Table 1 : Bianchi classification of three dimensional Lie algebras.

Type	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII_0	0	1	1	0
VI_0	0	1	-1	0
IX	0	1	1	1
$VIII$	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
VII_a	a	0	1	1
$III \quad (a=1)$ $VI_a \quad (a \neq 1)$	a	0	1	-1

Then by considering the dual Lie algebra in the form (17) and using the relation (16) we had obtained all Bianchi bialgebras. In ref [10] Hlavaty and Snobl, by considering dual algebras which are isomorphic to Bianchi algebras $\tilde{\mathbf{g}}$, have obtained (complete list) 44 real three dimensional Lie bialgebras. These isomorphism must be such that the ad-invariant metric (12) remains invariant under this transformations i.e:

$$\tilde{X}'^j = A^j{}_k \tilde{X}^k \qquad , \qquad X'_i = X_k (A^{-1})^k{}_i. \tag{18}$$

Their list of 44 Lie bialgebras contain 19 Lie bialgebras of our list in [9], with the names $(\mathbf{g}, I)$, $(VII_a, II) = (VII_a, II.i)$, $(VII_o, V) = (VII_o, V.i)$, (VI_a, II), (VI_o, II), $(VI_o, V) = (VI_o, V.i)$, $(V, II) = (V, II.i)$, $(IV, II) = (IV, II.i)$ and (III, II). Notice that these 44 Lie bialgebras are non-isomorphic. For the 28 Lie bialgebras that we have previously obtained, it is trivial. For the other pair of Lie bialgebras such as $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ where $\tilde{\mathbf{g}} \cong \tilde{\mathbf{g}}'$, as we have previously mentioned for investigation of the Lie bialgebra isomorphism, we must examine if relation (3) holds or not. By using (15) we can rewrite relation (3) as follows:

$$O_j{}^i \tilde{\mathcal{Y}}'_i = O^t \tilde{\mathcal{Y}}_j O, \tag{19}$$

where $(\tilde{\mathcal{Y}}_i)^{jk} = -\tilde{f}^{jk}{}_i$; $(\tilde{\mathcal{Y}}'_i)^{jk} = -\tilde{f}'^{jk}{}_i$ and we apply the matrix representation of the automorphism of the algebra $\mathbf{g}$ as follows:

$$O(X_i) = O_i{}^j X_j. \tag{20}$$

In this manner for investigation of isomorphism of such Lie bialgebras we must first obtain the automorphism groups of Bianchi algebras.

The automorphism groups of the complex three dimensional solvable Lie algebras were found previously in [7]. Here we find the automorphism groups of Bianchi algebras. These are Lie subgroups of $GL(3,R)$ which preserve the Lie brackets i.e:[2]

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad\qquad [X'_l, X'_m] = f_{lm}{}^n X'_n, \tag{21}$$

where by applying $X'_j = O_j{}^i X_i$ we have:

$$O_j{}^i O \mathcal{X}_i = \mathcal{X}_j O, \tag{22}$$

or,

$$\mathcal{Y}^j O_j{}^i = O \mathcal{Y}^i O^t, \tag{23}$$

where $(\mathcal{X}_i)_l{}^j = -f_{i\,l}{}^j$ are the adjoint representations of the bases of algebra $\mathbf{g}$ and as we mentioned above $(\mathcal{Y}^i)_{jk} = -f_{jk}{}^i$ are the antisymmetric matrices. Now we must first find the $\mathcal{X}_i$ or $\mathcal{Y}^i$ matrices for all Lie bialgebras. In [9], we have obtained general formulas for the matrices $\mathcal{X}_i$ and $\mathcal{Y}^i$. Now by knowing these matrices and applying relations (22) or (23) one can caculate general form of the elements of the automorphism groups of the Bianchi algebras. We have found and listed these in table 2:

Table 2 : Automorphism groups of the Bianchi algebras.

$\mathbf{g}$	Automorphism group	
I	$GL(3,R)$	
II	$\begin{pmatrix} detA & 0 \\ v & A \end{pmatrix}$	where $A \in GL(2,\Re), v \in \Re^2$
VII_0	$\begin{pmatrix} -c & d & 0 \\ d & c & 0 \\ v^t & & 1 \end{pmatrix}$	$c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
VI_0	$\begin{pmatrix} c & -d & 0 \\ d & -c & 0 \\ v^t & & 1 \end{pmatrix}$	$c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
IX	$SO(3)$	
$VIII$	$SL(2,R)$	
III, VI_a	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & d & c \end{pmatrix}$	$c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
V	$\begin{pmatrix} 1 & & v^t \\ 0 & & \\ 0 & & A \end{pmatrix}$	where $A \in GL(2,R)$ and $v \in \Re^2$
IV	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & 0 & c \end{pmatrix}$	$c, d \in \Re$ where $c \neq 0$ and $v \in \Re^2$
VII_a	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & -d & c \end{pmatrix}$	$c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$

[2]Notice that these are outer automorphism groups.

Now by knowing these automorphism groups we can investigate isomorphism of the pairs of Lie bialgebras of the form $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ by using relations (19). Note that the matrices $\tilde{\mathcal{X}}^i$ and $\tilde{\mathcal{Y}}_i$ have the same form as $\mathcal{X}_i$ and $\mathcal{Y}^i$ but we must replace the set(a, n_1, n_2, n_3) with $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$.
These matrices can be applied for the 19 Lie bialgebras mentioned above. For the remaining 25 Lie bialgebras one can obtain these matrices. Note that for these Lie bialgebras the matrices $\mathcal{X}_i$ and $\mathcal{Y}^i$ can be obtained from equations (15) of ref [9]; for this reason one can obtain only the matrices $\tilde{\mathcal{X}}^i$ and $\tilde{\mathcal{Y}}_i$ of these Lie bialgebras, for example for Lie bialgebra $(IX, V|b)$ we have:

$$\tilde{\mathcal{X}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \tilde{\mathcal{X}}_2 = \begin{pmatrix} 0 & -b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\mathcal{X}}_3 = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{Y}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\mathcal{Y}}_2 = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\mathcal{Y}}_3 = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}.$$

In this manner, we investigate the isomorphicity and find that the relations (19) do not satisfy for the pair of Lie bialgebras of the form $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ mentioned in [10]. For example, for the Lie bialgebras $(VIII, V.i|b)$ and $(VIII, V.ii|b)$ relation (19) for $j = 1$ does not satisfy and so on ... . Now we are ready to determine how many of the 44 real three dimensional Lie bialgebras are coboundary?

3 Three dimensional real coboundary Lie bialgebras

In this section we determine how many of 44 Lie bialgebras are coboundary? Therefore, we must find $r = r^{ij} X_i \otimes X_j \in \mathbf{g} \otimes \mathbf{g}$ such that the cocommutator of Lie bialgebras can be written as (4). By use of (4), (13) and (15) we have:

$$\tilde{\mathcal{Y}}_i = \mathcal{X}_i{}^t r + r \mathcal{X}_i. \tag{24}$$

Now by using of (24) and form of $\mathcal{X}$, $\mathcal{Y}$ matrices we can find the r-matrix of the Lie bialgebras. In this manner we determine which of the Lie bialgebras are coboundary and obtain r-matrices. Of course we also perform this work for the dual Lie bialgebras $(\tilde{\mathbf{g}}, \mathbf{g})$ by using the following equations as the same as (24):

$$\mathcal{Y}^i = (\tilde{\mathcal{X}}^i)^t \tilde{r} + \tilde{r} \tilde{\mathcal{X}}^i, \tag{25}$$

where as above $(\tilde{\mathcal{X}}^i)_l{}^j = -\tilde{f}^{ij}{}_l$ are the adjoint representations of the bases of algebra $\tilde{\mathbf{g}}$. The results are summerized in the following tables 3 and 4. Notice that, we also determine the Schouten brackets of the Lie bialgebras. In this manner the type of Lie bialgebras(triangular or quasi-triangular) are specified and we classify all three dimensional real coboundary Lie bialgebras. There are two points in this tables. First, we have listed coboundary Lie bialgebras with coboundary duals in a separate table 4. Since such structures can be specified (up to automorphism) by pairs of r-matrices, then it is natural to call them bi-r-matrix bialgebras

(b-r-b)[15][3]. In [15], some examples of three dimensional b-r-b have been given. Here we give complete list of three dimensional b-r-b. Secondly, as it is seen we have considered skew symmetric r-matrix solutions in tables 3 and 4. Of course there are other solutions for some Lie bialgebras of these tables. We have listed these solutions in other table $3'$. In this table Lie bialgebras (III, I), (VI_a, I) and $(VIII, V.i|b)$ are factorizable Lie bialgebras. Other Lie bialgebras of this table are quasitriangular such as they having r-matrix solutions with invariant symmetric part, which for special case ($c = d = e = 0$) transform to triangular solutions of tables 3 and 4.

Notice that in the following tables c, d and e are arbitrary nonzero constants.

Table 3 : Three dimensional coboundary Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	r	$[[r, r]]$
(II, I)	$cX_1 \wedge X_2 + dX_1 \wedge X_3$	0
(VII_o, I)	$cX_1 \wedge X_2$	0
$(VII_o, V.i)$	$X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$
(VI_0, I)	$cX_1 \wedge X_2$	0
$(VI_0, V.i)$	$X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$
$(IX, V\|b)$	$bX_2 \wedge X_3$	$b^2 X_1 \wedge X_2 \wedge X_3$
$(VIII, V.i\|b)$	$bX_2 \wedge X_3$	$b^2 X_1 \wedge X_2 \wedge X_3$
$(VIII, V.ii\|b)$	$-bX_1 \wedge X_2$	$-b^2 X_1 \wedge X_2 \wedge X_3$
$(VIII, V.iii)$	$-X_1 \wedge X_2 - X_2 \wedge X_3$	0
$(IV, II.i)$	$-X_2 \wedge X_3$	0
$(IV, II.ii)$	$\frac{1}{2} X_2 \wedge X_3$	0
$(IV.ii, VI_o)$	$\frac{1}{2}(X_1 \wedge X_3 + X_2 \wedge X_3)$	0
$(VII_a, II.i)$	$-\frac{1}{2a} X_2 \wedge X_3$	0
$(VII_a, II.ii)$	$\frac{1}{2a} X_2 \wedge X_3$	0
(III, II)	$-\frac{1}{2} X_2 \wedge X_3$	0
(VI_a, II)	$-\frac{1}{2a} X_2 \wedge X_3$	0

Table 4 : Three dimensional bi-r-matrix bialgebras.

$\mathbf{g}$	r	$[[r, r]]$	$\tilde{\mathbf{g}}$	$\tilde{r}$	$[[\tilde{r}, \tilde{r}]]$
$II.i$	$cX_1 \wedge X_2 + dX_3 \wedge X_1 + X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$	V	$-\frac{1}{2} X_2 \wedge X_3$	0
VI_o	$cX_1 \wedge X_2 - X_2 \wedge X_3 + X_3 \wedge X_1$	0	$V.ii$	$\frac{1}{2}(X_1 \wedge X_3 + X_2 \wedge X_3)$	0
III	$-\frac{1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0	$III.ii$	$X_1 \wedge X_2 + X_3 \wedge X_1$	0
III	$-\frac{1}{2}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0	$III.iii$	$X_1 \wedge X_2 + X_1 \wedge X_3$	0
VI_a	$-\frac{1}{a-1}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0	$VI_{\frac{1}{a}}.ii$	$\frac{a-1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0
VI_a	$-\frac{1}{a+1}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0	$VI_{\frac{1}{a}}.iii$	$\frac{a+1}{2}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0

[3]The most interesting applications of b-r-b are possible in the theory of bi-Hamiltonian dynamical systems [17]. In this case, the presence of pair of r-matrices allow us to define the pair of dynamical systems on the space which is the space of original Lie algebras canonically identified with its dual space [2].

Table $3'$: Three dimensional coboundary Lie bialgebras (other solutions).

$(\mathbf{g}, \tilde{\mathbf{g}})$	r	$[[r,r]]$
(III, II)	$cX_2 \bigotimes X_2 - (c+\frac{1}{2})X_2 \bigotimes X_3 - (c-\frac{1}{2})X_3 \bigotimes X_2 + cX_3 \bigotimes X_3$	0
(II, I)	$eX_1 \bigotimes X_1 + cX_1 \wedge X_2 + dX_1 \wedge X_3$	0
(III, I)	$c(-X_2 \bigotimes X_2 + X_2 \bigotimes X_3 + X_3 \bigotimes X_2 - X_3 \bigotimes X_3)$	0
(VI_a, I)	$c(X_2 \bigotimes X_2 + X_2 \bigotimes X_3 + X_3 \bigotimes X_2 + X_3 \bigotimes X_3)$	0
$(VIII, V.i\|b)$	$bX_2 \wedge X_3 \pm b(X_1 \bigotimes X_1 + X_2 \bigotimes X_2 - X_3 \bigotimes X_3)$	0
$(VI_o, V.ii)$	$d(X_1 \bigotimes X_1 - X_2 \bigotimes X_2) + cX_1 \wedge X_2 - X_2 \wedge X_3 + X_3 \wedge X_1$	0
$(III, III.ii)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) - \frac{1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0
$(III.ii, III)$	$cX_1 \bigotimes X_1 + X_1 \wedge X_2 + X_3 \wedge X_1$	0
$(III, III.iii)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) - \frac{1}{2}(X_1 \wedge X_2 - X_3 \wedge X_1)$	0
$(III.iii, III)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) + X_1 \wedge X_2 - X_3 \wedge X_1$	0

Notice that these coboundary Lie bialgebras are non-isomorphic. In the previous section we mentioned to the conditions (relation (5)) under which the coboundary Lie bialgebras are isomorphic. Here we consider this conditions in a more exact way and not formal. By using the matrix form of the isomorphism map $\alpha : \mathbf{g} \longrightarrow \mathbf{g}'$ i.e:

$$\alpha(X_i) = \alpha_i^{\ j} X'_j, \tag{26}$$

then relation (5) can be rewrite as :

$$\mathcal{X}'^{\,t}_i(\alpha^t r \alpha - r') = (\mathcal{X}'^{\,t}_i(\alpha^t r \alpha - r'))^t, \tag{27}$$

i.e, if the above matrices are symmetric then the two coboundary Lie bialgebras $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}', \tilde{\mathbf{g}}')$ are isomorphic. Note that for some pair of Lie bialgebras the matrix α is the same of the matrix A which we have previously mensioned in (18) and for some other pairs it is the combination of two A matrices. To find of the matrices A one can use the relation (18) and the following ones:

$$[X_i, X_j] = f_{ij}^{\ \ k} X_k, \qquad\qquad [X'_l, X'_m] = f'_{lm}{}^{n} X'_n. \tag{28}$$

Then one finds the following equation for the matrix A:

$$A\tilde{\mathcal{Y}}_j A^t = \tilde{\mathcal{Y}}'_i A^i_{\ j}, \tag{29}$$

by using this relations one can find the A matrices. We perform this works and find A and then α matrices for the pair of some Lie bialgebras, they are listed in appendix. By using this matrices we have found that the matrices (27) are non-symmetric; in other words all coboundary Lie bialgebras of tables 3 and 4 are non-isomorphic. For example note the Lie bialgebras $(V.ii, VI_o)$ and $(V, II.i)$ then by using (3) of appendix for the matrix A one can see that the relation (27) do not satisfy.

Notice that one can not completely compare our results with the results of [11]. In [11], the author has applied the classification of three dimensional Lie algebras that mentioned in [12]. Hence our results are not completely consistent with the results [11]. For the algebra $SO(3) = IX$ our results are compatible with the results of [11], because this Lie algebras are

the same; but for other Lie algebras, because of isomorphicity of algebras with the Bianchi ones, the results are not exactly the same as in [11].

Before beginning the next section let us discuses some about the application of classical r-matrix in the integrable systems. Indeed one can construct integrable systems over the vector space $\mathbf{g}^*$ related to the quasitriangular Lie bialgebras $(\mathbf{g}, \tilde{\mathbf{g}})$. One can perform this by using of the following proposition [3]:

Proposition: Let H be a smooth function on $\mathbf{g}^*$ which is invariant under coadjoint action of $\mathbf{G}$ (Lie group of $\mathbf{g}$) and let $r \in \mathbf{g} \otimes \mathbf{g}$ be a skew-symmetric solution of the modified CYBE. Then, the Hamiltonian system on $\mathbf{g}^*$ with Poisson bracket $\{,\}_r$ and Hamiltonian H admits a Lax pair (L, P). Moreover

$$\{L, L\}_r = [r, L \otimes 1 + 1 \otimes L]. \tag{30}$$

Where $\{,\}_r$ is the Poisson structure related to the following Lie bracket over $\mathbf{g}$:

$$[X, Y]_r = [\rho(X), Y] + [X, \rho(Y)] \tag{31}$$

where $\rho : \mathbf{g} \to \mathbf{g}$ is a linear map such that:

$$\rho(X_i) = \sum_j r^{ij} X_j \tag{32}$$

$L : \mathbf{g}^* \to \mathbf{g}$ is a canonical map with $L(\xi) = (\xi \otimes 1)(t)$ where $t \in \mathbf{g} \otimes \mathbf{g}$ is the casimir element and $P(\xi) = \rho(dH(\xi)) \qquad \forall \xi \in \mathbf{g}^*$.

Now by using of this proposition one can construct integrable systems related to the three dimensional quasitriangular Lie bialgebras. For example one can see that integrable system over the vector space $V.i$ related to the Lie bialgebras $(VIII, V.i|b)$ is the toda system with potential $\exp 2bq$.

4 Calculation of Poisson structures by Sklyanin bracket

We know that for the triangular and quasitriangular Lie bialgebras one can obtain their corresponding Poisson-Lie groups by means of the Sklyanin bracket provided by a given skew-symmetric r-matrix $r = r^{ij} X_i \wedge X_j$ [3]:

$$\{f_1, f_2\} = \sum_{i,j} r^{ij}((X_i^L f_1)\,(X_j^L f_2) - (X_i^R f_1)\,(X_j^R f_2)) \qquad \forall f_1, f_2 \in C^\infty(G) \tag{33}$$

where X_i^L and X_i^R are left and right invariant vector fields on the three dimensional related Lie group G. In the case that r is a solution of (CYBE), the following brackets are also Poisson structures on the group G:

$$\{f_1, f_2\}^L = \sum_{i,j} r^{ij}((X_i^L f_1)\,(X_j^L f_2) \tag{34}$$

$$\{f_1, f_2\}^R = \sum_{i,j} r^{ij}((X_i^R f_1)\,(X_j^R f_2) \tag{35}$$

To calculate the left and right invariant vector fields on the group G it is enough to determine the left and right one forms. For $g \in G$ we have:

$$dgg^{-1} = R^i X_i \qquad (dgg^{-1})^i = R^i = R^i{}_j dx^j, \tag{36}$$

$$g^{-1}dg = L^i X_i \qquad (g^{-1}dg)^i = L^i = L^i{}_j dx^j, \tag{37}$$

where x^i are parameters of the group spaces. Now from $\delta_j{}^i =< X_j^R, R^i >$ and $\delta_j{}^i =< X_j^L, L^i >$ where $X_j^R = X^R{}_j{}^l \partial_l$ and $X_j^L = X^L{}_j{}^l \partial_l$, we obtain:

$$X^R{}_j{}^l = (R^{-t})_j{}^l, \qquad X^L{}_j{}^l = (L^{-t})_j^l \tag{38}$$

To calculate the above matrices we assume the following parameterization of the group G:

$$g = e^{x_1 X_1} e^{x_2 X_2} e^{x_3 X_3}. \tag{39}$$

Then, in general, for left and right invariant Lie algebra valued one forms we have:

$$dgg^{-1} = dx_1 X_1 + dx_2 e^{x_1 X_1} X_2 e^{-x_1 X_1} + dx_3 e^{x_1 X_1}(e^{x_2 X_2} X_3 e^{-x_2 X_2})e^{-x_1 X_1}, \tag{40}$$

$$g^{-1}dg = dx_1 e^{-x_3 X_3}(e^{-x_2 X_2} X_1 e^{x_2 X_2})e^{x_3 X_3} + dx_2 e^{-x_3 X_3} X_2 e^{x_3 X_3} + dx_3 X_3. \tag{41}$$

As it is seen in the above calculations we need to calculate expressions such as $e^{-x_i X_i} X_j e^{x_i X_i}$ [4]. Indeed in [9] we have shown that:

$$e^{-x_i X_i} X_j e^{x_i X_i} = (e^{x_i \mathcal{X}_i})_j{}^k X_k, \tag{42}$$

where summation over index k is assumed.

For Bianchi algebras the form of matrices $e^{x_i \mathcal{X}_i}$ are obtained in [9]. For other Lie algebras which are isomorphic to the Bianchi ones we must calculate these matrices directly from the form of $\mathcal{X}_i$. We have performed these calculations only for Lie algebras $\mathbf{g}$ of $(\mathbf{g}, \tilde{\mathbf{g}})$ coboundary Lie bialgebras and then have obtained left and right invariant vector fields as given in table 5:

[4] Notice that repeated indices do not imply summation.

Table 5.1 : left and right invariant vector fields over 3-dimensional coboundary Bianchi groups.

g	$\begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix}$	$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix}$
$II.i$	$\begin{pmatrix} \partial_1 \\ -x_3\partial_1+\partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1+\partial_3 \end{pmatrix}$
VII_o	$\begin{pmatrix} \cos x_3\partial_1+\sin x_3\partial_2 \\ -\sin x_3\partial_1+\cos x_3\partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1+x_1\partial_2+\partial_3 \end{pmatrix}$
VI_0	$\begin{pmatrix} \cosh x_3\partial_1-\sinh x_3\partial_2 \\ -\sinh x_3\partial_1+\cosh x_3\partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1-x_1\partial_2+\partial_3 \end{pmatrix}$
IX	$\begin{pmatrix} \frac{\cos x_3}{\cos x_2}\partial_1+\sin x_3\partial_2-\tan x_2\cos x_3\partial_3 \\ \frac{-\sin x_3}{\cos x_2}\partial_1+\cos x_3\partial_2+\tan x_2\sin x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \tan x_2\sin x_1\partial_1+\cos x_1\partial_2-\frac{\sin x_1}{\cos x_2}\partial_3 \\ -\tan x_2\cos x_1\partial_1+\sin x_1\partial_2+\frac{\cos x_1}{\cos x_2}\partial_3 \end{pmatrix}$
$VIII$	$\begin{pmatrix} \frac{\cos x_3}{\cos x_2}\partial_1+\sin x_3\partial_2-\tanh x_2\cos x_3\partial_3 \\ \frac{\sin x_3}{\cosh x_2}\partial_1+\cos x_3\partial_2+\tanh x_2\sin x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \tanh x_2\sin x_1\partial_1+\cosh x_1\partial_2+\frac{\sinh x_1}{\cosh x_2}\partial_3 \\ -\tanh x_2\cosh x_1\partial_1+\sinh x_1\partial_2+\frac{\cosh x_1}{\cosh x_2}\partial_3 \end{pmatrix}$
V	$\begin{pmatrix} \partial_1+x_2\partial_2+x_3\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{x_1}\partial_2 \\ e^{x_1}\partial_3 \end{pmatrix}$
$V.ii$	$\begin{pmatrix} e^{x_2}\partial_1+(1-e^{x_2})\partial_2+(e^{x_2}(x_3-1)-x_3)\partial_3 \\ \partial_2-x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ (1-e^{-x_1})\partial_1+e^{-x_1}\partial_2 \\ e^{-x_1-x_2}\partial_3 \end{pmatrix}$
IV	$\begin{pmatrix} \partial_1+x_2\partial_2+(x_3-x_2)\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{x_1}\partial_1-x_1e^{x_1}\partial_2 \\ e^{x_1}\partial_3 \end{pmatrix}$
$IV.ii$	$\begin{pmatrix} e^{x_2}\partial_1+(1-e^{x_2})\partial_2-(x_2+x_3)\partial_3 \\ \partial_2-x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ (1-e^{-x_1})\partial_1+e^{-x_1}\partial_2-x_1e^{-x_1}\partial_3 \\ e^{-x_1}\partial_3 \end{pmatrix}$
VII_a	$\begin{pmatrix} \partial_1+(ax_2+x_3)\partial_2+(ax_3-x_2)\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{ax_1}\cos x_1\partial_2-e^{ax_1}\sin x_1\partial_3 \\ e^{ax_1}\sin x_1\partial_2+e^{ax_1}\cos x_1\partial_3 \end{pmatrix}$
III	$\begin{pmatrix} \partial_1+(x_2+x_3)(\partial_2+\partial_3) \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \frac{1+e^{2x_1}}{2}\partial_2+\frac{e^{2x_1}-1}{2}\partial_3 \\ \frac{e^{2x_1}-1}{2}\partial_2+\frac{1+e^{2x_1}}{2}\partial_3 \end{pmatrix}$
$III.ii$	$\begin{pmatrix} \partial_1 \\ e^{-x_3}\partial_2+(e^{-x_3}-1)\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ (e^{-x_2}-1)\partial_2+e^{-x_2}\partial_3 \end{pmatrix}$
$III.iii$	$\begin{pmatrix} e^{-x_2-x_3}\partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1+\partial_2 \\ -x_1\partial_1+\partial_3 \end{pmatrix}$

Table 5.2 : left and right invariant vector fields over 3-dimensional coboundary Bianchi groups(continue).

$\mathbf{g}$	$\begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix}$	$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix}$
VI_a	$\begin{pmatrix} \partial_1 + (ax_2 + x_3)(\partial_2 + \partial_3) \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{ax_1}(\cosh x_1 \partial_2 + \sinh x_1 \partial_3) \\ e^{ax_1}(\sinh x_1 \partial_2 + \cosh x_1 \partial_3) \end{pmatrix}$
$VI_{\frac{1}{a}}.ii$	$\begin{pmatrix} e^{x_3 - x_2}\partial_1 \\ e^{-\alpha x_3}\partial_2 + (e^{-\alpha x_3} - 1)\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ x_1\partial_1 + (e^{-\alpha x_2} - 1)\partial_2 + e^{-\alpha x_2}\partial_3 \end{pmatrix}$, $\alpha = \frac{a+1}{a-1}$
$VI_{\frac{1}{a}}.iii$	$\begin{pmatrix} e^{-x_2 - x_3}\partial_1 \\ e^{\frac{1}{\alpha}x_3}\partial_2 + (1 - e^{\frac{1}{\alpha}x_3})\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ -x_1\partial_1 + (1 - e^{-\frac{1}{\alpha}x_2})\partial_2 + e^{-\frac{1}{\alpha}x_2}\partial_3 \end{pmatrix}$, $\alpha = \frac{a+1}{a-1}$

Now by using these results we can calculate the Poisson structures over the group G. For simplicity we can rewrite relation (33) in the following matrix form:

$$\{f_1, f_2\} = \begin{pmatrix} X_1^L f_1 & X_2^L f_1 & X_3^L f_1 \end{pmatrix} r \begin{pmatrix} X_1^L f_2 \\ X_2^L f_2 \\ X_3^L f_2 \end{pmatrix} - \begin{pmatrix} X_1^R f_1 & X_2^R f_1 & X_3^R f_1 \end{pmatrix} r \begin{pmatrix} X_1^R f_2 \\ X_2^R f_2 \\ X_3^R f_2 \end{pmatrix}, \tag{43}$$

and similarly we can rewrite (34) and (35).

In this manner, we calculate the fundamental Poisson brackets of all triangular and quasitriangular Lie bialgebras. The results are given in tables 6 and 7. Notice that for triangular Lie bialgebras we have calculated all Poisson structures (33), (34) and (35) and have listed in the separate table 7.

Table 6 : Poisson brackets related to the quasi-triangular Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$(II.i, V)$	$-x_2$	$-x_3$	0
$(VII_o, V.i)$	$-x_2$	$-\sin x_3$	$\cos x_3 - 1$
$(VI_o, v.i)$	$-x_2$	$-\sinh x_3$	$\cosh x_3 - 1$
$(IX, V\|b)$	$-b\tan x_2$	$-b\frac{\sin x_3}{\cos x_2}$	$b(\cos x_3 - \frac{1}{\cos x_2})$
$(VIII, V.i\|b)$	$-b\tanh x_2(2\cosh^2 x_1 - 1)$	$b\frac{\sin x_3 - \tanh x_2 \sinh 2x_1}{\cosh x_2}$	$b(\cos x_3 - \frac{1}{\cosh x_2})$
$(VIII, V.ii\|b)$	$b\frac{-\cos 2x_3 + \cosh x_1 \cosh x_2}{\cosh x_2}$	$b\frac{\sinh x_1 - \tanh x_2 \sin 2x_3}{\cosh x_2}$	$-b\tanh x_2$

Table 7.1 : Poisson brackets related to some triangular Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	(II, I)	(VII_o, I)	(VI_o, I)	$(VI_o, V.ii)$	$(V, II.i)$	$(V.ii, VI_o)$
$\{x_1, x_2\}^L$	c	c	c	c	0	0
$\{x_1, x_3\}^L$	c'	0	0	$\sinh x_3 - \cosh x_3$	0	$\frac{e^{x_2}}{2}$
$\{x_2, x_3\}^L$	0	0	0	$\sinh x_3 - \cosh x_3$	$-\frac{1}{2}$	$1 - \frac{e^{x_2}}{2}$
$\{x_1, x_2\}^R$	c	c	c	$c - x_2 - x_1$	0	0
$\{x_1, x_3\}^R$	c'	0	0	-1	0	$e^{-x_1 - x_2}(1 - e^{-x_1})$
$\{x_2, x_3\}^R$	0	0	0	-1	$-\frac{e^{2x_1}}{2}$	$\frac{e^{-2x_1 - x_2}}{2}$
$\{x_1, x_2\}$	0	0	0	$x_2 + x_1$	0	0
$\{x_1, x_3\}$	0	0	0	$\sinh x_3 - \cosh x_3 + 1$	0	$\frac{e^{x_2}}{2} - e^{-x_1 - x_2}(1 - e^{-x_1})$
$\{x_2, x_3\}$	0	0	0	$\sinh x_3 - \cosh x_3 + 1$	$\frac{e^{2x_1} - 1}{2}$	$1 - \frac{e^{x_2} + e^{-2x_1 - x_2}}{2}$

Table 7.2 : Poisson brackets related to triangular Lie bialgebras (continue).

$(\mathbf{g}, \tilde{\mathbf{g}})$	$(VIII, V.iii)$	$(IV, II.i)$	$(IV, II.ii)$	$(IV.ii, VI_o)$
$\{x_1, x_2\}^L$	$-\frac{\cos 2x_3}{\cosh x_2}$	0	0	0
$\{x_1, x_3\}^L$	$-\frac{\sin x_3(2\tanh x_2 \cos x_3+1)}{\cosh x_2}$	0	0	$\frac{e^{x_2}}{2}$
$\{x_2, x_3\}^L$	$-\tanh x_2 - \cos x_3$	-1	$\frac{1}{2}$	$1-\frac{e^{x_2}}{2}$
$\{x_1, x_2\}^R$	$-\cosh x_1 - \tanh x_2 \cosh 2x_1$	0	0	0
$\{x_1, x_3\}^R$	$-\frac{\sinh x_1(2\tanh x_2 \cosh x_1+1)}{\cosh x_2}$	$-e^{2x_1}$	$\frac{e^{x_1}}{2}$	$\frac{e^{-x_1}(2-e^{-x_1})}{2}$
$\{x_2, x_3\}^R$	$-\frac{1}{\cosh x_2}$	$x_1 e^{2x_1}$	$-\frac{x_1 e^{2x_1}}{2}$	$\frac{e^{-2x_1}}{2}$
$\{x_1, x_2\}$	$-\frac{\cos 2x_3}{\cosh x_2} + \cosh x_1 + \tanh x_2 \cosh 2x_1$	0	0	0
$\{x_1, x_3\}$	$-\frac{\sin x_3(2\tanh x_2 \cos x_3+1)-\sinh x_1(2\tanh x_2 \cosh x_1+1)}{\cosh x_2}$	e^{2x_1}	$-\frac{e^{2x_1}}{2}$	$\frac{e^{x_2}+e^{-x_1}(e^{-x_1}-2)}{2}$
$\{x_2, x_3\}$	$-\tanh x_2 - \cos x_3 + \frac{1}{\cosh x_2}$	$-1 - x_1 e^{2x_1}$	$\frac{1+x_1 e^{2x_1}}{2}$	$1-\frac{e^{x_2}+e^{-2x_1}}{2}$

Table 7.3 : Poisson brackets related to triangular Lie bialgebras (continue).

$(\mathbf{g}, \tilde{\mathbf{g}})$	(III, II)	$(III, III.ii)$	$(III, III.iii)$	$(III.ii, III)$	$(III.iii, III)$
$\{x_1, x_2\}^L$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	e^{-x_3}	$e^{-x_2-x_3}$
$\{x_1, x_3\}^L$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$e^{-x_3}-2$	$e^{-x_2-x_3}$
$\{x_2, x_3\}^L$	$-\frac{1}{2}$	$x_2 + x_3$	0	0	0
$\{x_1, x_2\}^R$	0	$-\frac{1}{2}$	$-\frac{e^{2x_1}}{2}$	$2-e^{-x_2}$	1
$\{x_1, x_3\}^R$	0	$\frac{1}{2}$	$-\frac{e^{2x_1}}{2}$	$-e^{-x_2}$	1
$\{x_2, x_3\}^R$	$-\frac{e^{2x_1}}{2}$	0	0	0	0
$\{x_1, x_2\}$	0	0	$\frac{e^{2x_1}-1}{2}$	$e^{-x_2}+e^{-x_3}-2$	$e^{-x_2-x_3}-1$
$\{x_1, x_3\}$	0	0	$\frac{e^{2x_1}-1}{2}$	$e^{-x_2}+e^{-x_3}-2$	$e^{-x_2-x_3}-1$
$\{x_2, x_3\}$	$\frac{e^{2x_1}-1}{2}$	$x_2 + x_3$	0	0	0

Table 7.4 : Poisson brackets related to some triangular Lie bialgebras (continue).

$(\mathbf{g}, \tilde{\mathbf{g}})$	(VI_a, II)	$(VI_a, VI_{\frac{1}{a}}.ii)$	$(VI_a, VI_{\frac{1}{a}}.iii)$	$(VI_{\frac{1}{a}}.ii, VI_a)$
$\{x_1, x_2\}^L$	0	$-\frac{1}{a-1}$	$-\frac{1}{a+1}$	$\frac{a-1}{2} e^{-x_2+(1-\alpha)x_3}$
$\{x_1, x_3\}^L$	0	$\frac{1}{a-1}$	$-\frac{1}{a+1}$	$\frac{a-1}{2} e^{x_3-x_2}(e^{-\alpha x_3}-2)$
$\{x_2, x_3\}^L$	$-\frac{1}{2a}$	$\alpha(x_2+x_3)$	$\frac{x_3-x_2}{\alpha}$	0
$\{x_1, x_2\}^R$	0	$\frac{e^{ax_1}(\sinh x_1 - \cosh x_1)}{a-1}$	$-\frac{e^{\frac{x_1}{a}}(\sinh x_1 + \cosh x_1)}{a+1}$	$\frac{a-1}{2}(2-e^{-\alpha x_2})$
$\{x_1, x_3\}^R$	0	$-\frac{e^{ax_1}(\sinh x_1 - \cosh x_1)}{a-1}$	$-\frac{e^{\frac{x_1}{a}}(\sinh x_1 + \cosh x_1)}{a+1}$	$-\frac{a-1}{2} e^{-\alpha x_2}$
$\{x_2, x_3\}^R$	$-\frac{e^{2ax_1}}{2a}$	0	0	0
$\{x_1, x_2\}$	0	$-\frac{1+e^{ax_1}(\sinh x_1 - \cosh x_1)}{a-1}$	$\frac{-1+e^{\frac{x_1}{\alpha}}(\sinh x_1 + \cosh x_1)}{a+1}$	$\frac{a-1}{2}(e^{-x_2+(1-\alpha)x_3}+e^{-\alpha x_2}-2)$
$\{x_1, x_3\}$	0	$\frac{1+e^{ax_1}(\sinh x_1 - \cosh x_1)}{a-1}$	$\frac{-1+e^{\frac{x_1}{\alpha}}(\sinh x_1 + \cosh x_1)}{a+1}$	$\frac{a-1}{2}(e^{x_3-x_2}(e^{-\alpha x_3}-2)+e^{-\alpha x_2})$
$\{x_2, x_3\}$	$\frac{e^{2ax_1}-1}{2a}$	$\alpha(x_2+x_3)$	$\frac{x_3-x_2}{2}$	0

Table 7.5 : Poisson brackets related to some triangular Lie bialgebras (continue).

$(\mathbf{g},\tilde{\mathbf{g}})$	$(VI_{\frac{1}{a}}.iii, VI_a)$	$(VII_a, II.i)$	$(VII_a, II.ii)$
$\{x_1,x_2\}^L$	$\frac{a+1}{2}e^{-x_2-\frac{2x_3}{a+1}}$	0	0
$\{x_1,x_3\}^L$	$\frac{a+1}{2}(e^{-x_2-x_3}-e^{-x_2-\frac{2x_3}{a+1}})$	0	0
$\{x_2,x_3\}^L$	0	$-\frac{1}{2a}$	$\frac{1}{2a}$
$\{x_1,x_2\}^R$	$\frac{a+1}{2}(2-e^{-\frac{x_2}{\alpha}})$	0	0
$\{x_1,x_3\}^R$	$\frac{a+1}{2}e^{-\frac{x_2}{\alpha}}$	0	0
$\{x_2,x_3\}^R$	0	$-\frac{e^{2ax_1}}{2a}$	$\frac{e^{2ax_1}}{2a}$
$\{x_1,x_2\}$	$\frac{a+1}{2}(e^{-x_2-\frac{2x_3}{a+1}}+e^{\frac{-x_2}{\alpha}}-2)$	0	0
$\{x_1,x_3\}$	$\frac{a+1}{2}(e^{-x_2-x_3}-e^{-x_2-\frac{2x_3}{a+1}}+e^{\frac{-x_2}{\alpha}})$	0	0
$\{x_2,x_3\}$	0	$\frac{e^{2ax_1-1}}{2a}$	$-\frac{e^{2ax_1-1}}{2a}$

Now by knowing the Poisson structures of the Poisson-Lie groups one can construct dynamical systems over the symplectic leaves of this Poisson-Lie groups as a phase spaces. This can be down by using of the dressing action of $\mathbf{G}^*$ (Lie group of $\mathbf{g}^*$) on $\mathbf{G}$ which is a Poisson action whose orbits are exactly the symplectic leaves of $\mathbf{G}$ [3], [4].

5 Concluding remarks

As mentioned above by determining the types (triangular or quasitriangular) and obtaining r-matrices and Poisson-Lie structures of the real three dimensional Lie bialgebras one can construct integrable systems over the vector space $\mathbf{g}^*$; meanwhile one is now ready to perform the quantization of these Lie bialgebras. Furthermore, now one can obtain Poisson-Lie T-dual sigma models over three dimensional triangular Lie bialgebras [18]. Notice that in [18] only example $su(2)$ was considered. On the other hand one can investigate integrability under Poisson-Lie T duality by studying the Poisson-Lie T dual sigma models over three dimensional bi-r- matrix bialgebras.

Appendix

Here, we list α matrices which are applied in relations (27).

1-For the pairs $((IV,II.i)$, $(IV.ii,VI_o))$ and $((IV,II.ii)$, $(IV.ii,IV_o))$:

$$\alpha = A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2-For the pair (II,I) and $(II.i,V)$:

$$\alpha = A = I$$

3-For the pair $(V,II.i)$ and $(V.ii,VI_o)$:

$$\alpha = A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ b & 0 & 0 \end{pmatrix}$$

4-For the pairs $((III,III.ii)\ ,(III.ii,III))$, $((III,III.iii)\ ,(III.ii,III))$ and $((III,II)\ ,(III.ii,III))$:

$$\alpha=A=\begin{pmatrix}0 & -c & c\\ -\frac{1}{2} & d & d+e-f\\ \frac{1}{2} & e & f\end{pmatrix}$$

where $c,d,e,f\in\Re$.

5-For the pairs $((III,III.iii)\ ,\ (III.iii,III))$, $((III,III.ii)\ ,\ (III.iii,III))$ and $((III,II)\ ,\ (III.iii,III))$:

$$\alpha=A=\begin{pmatrix}0 & c & c\\ \frac{1}{2} & d & f+e-d\\ \frac{1}{2} & e & f\end{pmatrix}$$

where $c,d,e,f\in\Re$.

6-For the pair $(VI_{\frac{1}{a}}.ii,VI_a)$ and $(VI_{\frac{1}{a}}.iii,VI_a)$:

$$\alpha=A{-}1(VI_{\frac{1}{a}}\longrightarrow VI_{\frac{1}{a}}.ii)A(VI_{\frac{1}{a}}\longrightarrow VI_{\frac{1}{a}}.iii)$$

where:

$$\alpha=A(VI_{\frac{1}{a}}\longrightarrow VI_{\frac{1}{a}}.ii)=\begin{pmatrix}0 & c & -c\\ \frac{a}{1-a} & d & e\\ -\frac{a}{1-a} & f & d+e-f\end{pmatrix}$$

and

$$\alpha=A(VI_{\frac{1}{a}}\longrightarrow VI_{\frac{1}{a}}.iii)=\begin{pmatrix}0 & c' & -c'\\ \frac{a}{1-a} & d' & e'+f'-d'\\ \frac{a}{1+a} & f' & e'\end{pmatrix}$$

where $c,d,e,f\in\Re$ and similarly for prime parameter.

7-For the pair $(III.ii,III)$ and $(III.iii,III)$:

$$\alpha=A{-}1(III\longrightarrow III.ii)A(III\longrightarrow III.iii)$$

References

[1] V. G. Drinfeld. *"Quantum Groups"*. In; Proc. ICM, MSRI, Berkeley, 1986. p. 798.

[2] M. A. Semenov-Tian-Shasky."What is a classical r-matrix?" Funct. Anal. Appl.17(1983)259-272.

[3] V. Chari and A. Pressley."A Guide to Quantum Groups". Cambridge Univ. Press. 1994.

[4] Y. Kosmann-Schwarzbach."Lie bialgebras, Poisson-Lie groups and dressing transformations". in Y. Kosmann-Schwarzbach, B. Grammaticod and K. M. Tamizhmani(edu)"Integrabilty of nonlinear systems". Proceedings, Pondicherry, India. Springer Verlag (1996)104-170.

[5] C. Klimcik and P. Severa."Dual nonAbelian duality and Drinfeld double." Phys. Lett B 351(1995)455, hep-th/9502122. C. Klimcik." Poisson-Lie T-duality". Nucl. Phys. Proc. Suppl. 46(1996) 116, hep-th/9509095.

[6] A. A. Belavin and V. G. Drinfeld."Solution of the classical Yang-Baxter equation for simple Lie algebras". Funct. Anal. 16(1983)159-180.

[7] J.M. Figueroa-O'Farrill."N=2 structures on solvable Lie algebras: The c=9 classification".Commun. Math.Phys. 177(1996)129-156, hep-th/9412008.

[8] S. Zhang."Classical Yang-Baxter equation and low dimensional triangular Lie bialgebras". Phys.Lett A 246(1998)71-81 , math.QA/0311517.

[9] M.A. Jafarizadeh and A. Rezaei-Aghdam,"Poisson-Lie T-duality and Bianchi type algebras". Phys. Lett B458(1999)477-490, hep-th/9903152.

[10] L.Hlavaty and L. Snobl."Classification of 6-dimensional Manin triples". math.QA/0202209.

[11] X. Gomez."Classification of three-dimensional Lie bialgebras". J. Math. Phys. 41(2000) 4939-4956.

[12] J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus."Invariants of real low dimension Lie algebras". J. Math. Phys. 17 (1976) 986-994.

[13] S. E. Parkhomenko."Extended superconformal current algebras and finite dimensional Manin triples". Sov. Phys. JETP. 75(1992)1-3. E. Getzler."Manin triples and N=2 superconformal field theory". hep-th/9307041.

[14] R. Von Unge."Poisson-Lie T-plurality". JHEP 0207(2002)014, hep-th/0205245. L. Hlavaty and L. Snobl."Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models". JHEP 0405(2004)010 ,hep-th/0403164. L. Hlavaty and L. Snobl."Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models II:Nondiagonal metrics and dilaton puzzle". JHEP 0410(2004)045 , hep-th/0408126.

[15] M. A. Sokolov."Some examples of classical coboundary Lie bialgebras with coboundary duals". math.QA/0103115.

[16] M. Ryan and L. Shepley, *"Homogeneous Relativistic Cosmologies"*, Princeton Univ. Press, Princeton, 1975; M. A. H. MacCallum, *"Anisotropic and Inhomogeneous Relativistic Cosmologies"*, in *"The Early Universe: Reprints"* by E. W. Kolb and M. S. Turner, Addison-Wisley. N.Y. 1990; L. D. Landau and E. M. Lifshitz, *"The Classical Theory of Fields"*, Pergamon Press, 1987.

[17] F. Magri."A simple model of the integrable Hamiltonian equation" J. Math. Phys. 19 (1978)1156-1162.

[18] S. Majid and E. J. Beggs."Poisson-Lie T-duality for quasitriangular Lie bialgebras". math.QA/9906040.